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ON THE STIFFNESS PROPERTY OF MOTION

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V.N.SKIMEL'
(Kazan')

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It is well known that the axis of a rapidly rotating gyroscope is very little responsive to large perturbing forces, i.e., possesses stiffness [1]. It was noticed that under specific conditions the stiffness property is inherent in systems with gyroscopes [2]. Many mechanical systems possess a property skin in some sense to gyroscopic stiffness. In this paper stiffness is interpreted as a distinctive stability. Theorems establishing tests for stiffness, analogous to the theorems of Liapunov's direct method, are formulated. The stiffness property is described by using a separation of variables as is done in problems on stability with respect to a part of the variables (see [3,4], etc.). Individual questions on the stiffness of motion were examined in [5,6].

1. Basic definitions. The equations of motion of a mechanical system are written as

$$dy/dt = Y(t, y, g) \tag{1.1}$$

where $t \geqslant 0$ is time, y is an n-dimensional state vector of the system and g is a constant vector-valued physical parameter. The motions (partial solution of (1.1))

$$y = f(t, y_0, g) \tag{1.2}$$

from some family, satisfying the initial conditions: $y = y_0$ when $t = t_0$, are considered to be the unperturbed motions. It should be noted that the values of y_0 and of parameter g can be related by the existence conditions for the motions (1.2).

Setting $y = f + x$ in (1.1), we obtain the equation of perturbed motion

$$dx/dt = Y(t, f + x, g) - Y(t, f, g) \tag{1.3}$$

in which y_0 and g are parameters. Equation (1.3) is considered dependent upon parameters $a_1, \ldots, a_r$ essential to the motion stiffness problem and is written as

$$dx/dt = X(t, x, a), \quad X(t, 0, a) \equiv 0 \tag{1.4}$$
$$a = (a_1, \ldots, a_r)$$

We shall investigate the stiffness of the motion $x = 0$ with respect to a part of the variables x_α $(\alpha = 1, \ldots, m;\ m < n)$. We assume that the vector-valued function $X(t, x, a)$ in (1.4) is continuous and satisfies the uniqueness conditions for the solution in the domain

$$\Gamma_0 = \{x: |x_\alpha| < H, |x_\beta| < \infty \ (\alpha = 1, \ldots, m;\ \beta = m + 1, \ldots, n)\}, \quad t \geqslant t^* \tag{1.5}$$

for all $a \in D$, where D is some domain in the space of the parameters being examined. We denote the solution of (1.4) by $x(t, t_0, x_0, a)$ and in the space of variables $\{x_1, \ldots, x_n\}$: we introduce a parallelepiped defined by the inequalities

$$\Pi_\varepsilon = \{x : |x_\alpha| \leqslant \varepsilon_1,\ |x_\beta| \leqslant \varepsilon_2\}, \quad \Pi_\delta = \{x : |x_\alpha| \leqslant \delta_1 < \varepsilon_1 \quad (1.6)$$
$$|x_\beta| \leqslant \delta_2 < \varepsilon_2\}$$

the set of boundary points is denoted $\bar{\Pi} \setminus \Pi$ in what follows.

Definition 1. Motion $x = 0$ possesses stiffness with respect to variables x_α if for any numbers $\varepsilon_1 > 0$ and $\delta_2 > 0$ (the first can be arbitrarily small, the second, arbitrarily large) and for an instant $t_0 \geqslant t^*$ we can find a parameter $a^* \in D$ and numbers $\varepsilon_2 > 0$ and $\delta_1 > 0$ defining domain (1.6), for which $x(t, t_0, x_0, a^*) \in \Pi$ when $t \geqslant t_0$ if only $x_0 \in \Pi_\delta$. The stiffness is said to be uniform in t_0 if a^*, ε_2 and δ_1 do not depend on t_0

Definition 2. Motion $x = 0$ possesses strong stiffness with respect to variables x_α, and domain Π_δ lies in its domain of attraction, if it possesses stiffness with respect to these variables and, in addition, the condition

$$\lim_{t \to \infty} x(t, t_0, x_0, a^*) = 0, \quad x_0 \in \Pi_\delta$$

holds.

If in Definition 1 we set $\varepsilon_1 = \varepsilon_2 = \varepsilon$ and $\delta_1 = \delta_2 = \delta$, we obtain a property of the motion, close to the practical stability discussed in [7]. Using this terminology, we arrive at the following definition; we denote $\|x\| = \max(|x_i|)$.

Definition 3. Motion $x = 0$ possesses practical stability if for any $\varepsilon > 0$ and instant $t_0 \geqslant t^*$ we can find a parameter $a^* \in D$ and a number $\delta > 0$ for which $\|x(t, t_0, x_0, a^*)\| < \varepsilon$ when $t \geqslant t_0$ if only $\|x_0\| \leqslant \delta$.

A motion's stiffness and its stability (also with respect to a part of the variables) are, in general, independent properties of the motion.

Example 1. Stiffness of a gyroscope's axis in the Euler case. As is customary let $Oxyz$ be the principal axes of inertia at the fastening point, $A = B$ and C are the corresponding moments of inertia. To describe the motion we use the variables $\gamma_1, \gamma_2, \gamma_3, p, q$ and r, where the γ_i are the cosines of the angles formed by some fixed axis z_1 with the axes x, y and z, respectively. In the unperturbed motion we set

$$\gamma_1^\circ = \gamma_2^\circ = 0, \ \gamma_3^\circ = 1, \quad p^\circ = q^\circ = 0, \quad r^\circ = \omega = \text{const} \qquad (1.7)$$

In the perturbed motion we denote

$$\begin{gathered} \gamma_1 = \eta_1, \quad \gamma_2 = \eta_2, \quad \gamma_3 = 1 - \eta_3 \quad (\eta_3 \geqslant 0) \\ p = \xi_1, \quad q = \xi_2, \quad r = \omega + \xi_3, \quad (\xi_3 = \text{const}) \end{gathered} \qquad (1.8)$$

From the relations $\gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1$ we can deduce that

$$\eta_1^2 + \eta_2^2 + \eta_3^2 = 2\eta_3 \qquad (1.9)$$

We shall examine stiffness with respect to the variables η_α $(\alpha = 1, 2, 3)$, taking $a = \omega$. Turning to Definition 1 we note that regarding relation (1.9) it is convenient to use the domains (β=1,2,3)

$$\overline{Q}_\varepsilon = \{x:\ \eta_1^2 + \eta_2^2 \leqslant \varepsilon_1^2 < 1,\ \eta_3 \leqslant 1 - \sqrt{1-\varepsilon_1^2} < \varepsilon_1;\ |\xi_\beta| \leqslant \varepsilon_2\} \tag{1.10}$$

instead of (1.6). We specify the numbers $\varepsilon_1 < 1$ and δ_2 and we first consider the perturbed motions under the initial conditions

$$\eta_{\alpha 0} = 0, \quad |\xi_{\beta 0}| \leqslant \delta_2 \tag{1.11}$$

In the perturbed motion the gyroscope's axis z, which coincides with the axis z_1 at the initial instant, describes a circular cone around the vector L of the moment of momentum; the angle at the cone's apex is 2θ, where

$$\sin\theta = A\omega_{10} / L,\quad \omega_{10} = (\xi_{10}^2 + \xi_{20}^2)^{1/2},\quad L = (A^2\omega_{10}^2 + C^2r^2)^{1/2}$$

Hence $\cos 2\theta \leqslant \gamma_3(t) \leqslant 1$; consequently (see (1.8)), $\eta_3 \leqslant 1 - \cos 2\theta$ or

$$\eta_3 \leqslant 2\,(A\omega_{10} / L)^2 \tag{1.12}$$

We now select the magnitude $|\omega|$ to make the condition $\eta_1^2 + \eta_2^2 < \varepsilon_1^2$ hold for the perturbed motions from (1.11) being examined; then $\eta_3 < 1 - \sqrt{1-\varepsilon_1^2}$. Bearing (1.12) in mind, we set

$$2\,(A\omega_{10} / L)^2 < 1 - \sqrt{1-\varepsilon_1^2}$$

whence follows the inequality

$$|\omega| > \delta_2\,(1 + \sqrt{2}\,A\varepsilon_1 / C\,(1 - \sqrt{1-\varepsilon_1^2})) \tag{1.13}$$

From the constancy of the gyroscope's kinetic energy it follows that $|\xi_\beta| < \varepsilon_2$ if $\varepsilon_2 > \sqrt{2}\delta_2$. Thus the inequalities $\eta_1^2 + \eta_2^2 < \varepsilon_1^2$ and $|\xi_\beta| < \varepsilon_2$ hold for the perturbed motions with initial conditions (1.11) when $t > t_0$ if the parameters value satisfies (1.13). However, we can be persuaded that δ_1 exists, dependent on the parameter's value chosen by (1.13), so small that the inequality holds even for $\eta_{10}^2 + \eta_{20}^2 \leqslant \delta_1^2$ and $|\xi_{\beta 0}| \leqslant \delta_2$.

2. Application of the Liapunov function method to the motion stiffness problem. 1°. We examine real single-valued functions $v = v(t, x, b)$ which, in general, depend on a part $b = (a_1, \ldots, a_p)$, $p \leqslant r$, of the variables contained in (1.4). We assume that functions v have been defined and are continuous together with the derivatives $\partial v / \partial t$ and $\partial v / \partial x_s$ $(s = 1, \ldots n)$ in the domain

$$\Gamma = \{x : |x_\alpha| \leqslant h < H,\ |x_\beta| < \infty\} \subset \Gamma_0,\quad t \geqslant t^*,\quad b \in D_1 \tag{2.1}$$

Definition 4. Function $v(x, b)$ possesses property (A) with respect to x_α if for any $\varepsilon_1 \in (0, h)$ and $\delta_2 > 0$ we can find $b^* \in D_1$, $\varepsilon_2 > 0$ and $\delta_1 > 0$, depending on them for which

$$\inf\,[v(x, b^*) : x \in \overline{\Pi}_\varepsilon \setminus \Pi_\varepsilon] > \sup\,[v(x, b^*) : x \in \overline{\Pi}_\delta] \tag{2.2}$$

Function $v(t, x, b)$ possesses property (A) if in domain (2.1)

$$v(t, x, b) \geqslant w(x, b) \tag{2.3}$$

and for every ε_1, δ_2 and $t_0 \geqslant t^*$ there are b^*, ε_2 and δ_1 for which

$$\inf[w(x, b^*) : x \in \overline{\Pi}_\varepsilon \setminus \Pi_\varepsilon] > \sup[v(t_0, x, b^*) : x \in \overline{\Pi}_\delta] \tag{2.4}$$

Function $v(t, x, b)$ possesses property (A) uniformly in $t \in [t^*, \infty)$ if in domain (2.1)

$$W(x, b) \geqslant v(t, x, b) \geqslant w(x, b) \tag{2.5}$$

and for every ε_1 and δ_2 there are b^*, ε_2 and δ_1 for which

$$\inf[w(x, b^*) : x \in \overline{\Pi}_\varepsilon \setminus \Pi_\varepsilon] > \sup[W(x, b^*) : x \in \overline{\Pi}_\delta] \tag{2.6}$$

Having denoted $r_1 = \max(|x_\alpha|)$ and $r_2 = \max(|x_\beta|)$, we consider the domain $G_\varepsilon = \{x: 0 \leqslant r_1 \leqslant \varepsilon, 0 \leqslant r_2 < \infty\}$ and its boundary point set $R_\varepsilon = \{x: r_1 = \varepsilon, 0 \leqslant r_2 < \infty\}$ for some $\varepsilon \in (0, h)$.

Lemma. For the function $v(x, b)$ to possess property (A) with respect to x_α it is sufficient that the following conditions be fulfilled:

a) $v = v^*(x_\beta)$ if $x_\alpha = 0$;

b) for any arbitrarily small $\varepsilon > 0$ and large $M > 0$ there exists a parameter $b^* \in D_1$ for which $\inf v(x, b^*) > M$ on the set $x \in R_\varepsilon$;

c) $v(x, b) \to +\infty$ as $r_2 \to +\infty$ uniformly relative to x_α in domain G_ε.

Proof. Let us show that domains $\overline{\Pi}_\varepsilon$, and $\overline{\Pi}_\delta$ for which inequality (2.2) is fulfilled can be constructed for functions satisfying the Lemma's hypotheses. Indeed, let ε_1 and δ_2 be specified. Then

$$\sup[v(x, b) : x_\alpha = 0, \quad |x_\beta| \leqslant \delta_2] = M(\delta_2)$$

We select $b^* \in D_1$ as to have

$$\inf v(x, b^*) > M(\delta_2), \quad x \in R_{\varepsilon_1}$$

We can find $\varepsilon_2 > \delta_2$, large enough to make

$$\inf[v(x, b^*) : x \in \overline{\Pi}_\varepsilon \setminus \Pi_\varepsilon] > M(\delta_2)$$

Finally, because the function $v(x, b^*)$ is uniformly continuous, a sufficiently small δ_1 defining $\overline{\Pi}_\delta$ in (2.2) exists in the domain $\overline{\Pi}_\varepsilon$.

Function $v(t, x, b)$ obviously possesses property (A) if $v = v^*(t, x_\beta)$ when $x_\alpha = 0$ and a function $w(x, b)$ satisfying (2.3) and the Lemma's hypotheses exists. If functions $W(x, b)$ and $w(x, b)$ satisfying (2.5) and the Lemma's hypotheses exist, then function $v(t, x, b)$ possesses property (A) uniformly in $t \in [t^*, \infty)$.

Note 1. We set $\varepsilon_1 = \varepsilon_2 = \varepsilon$ and $\delta_1 = \delta_2 = \delta$. We say that a function $v(x, b)$ that for this case satisfies condition (2.2) in Definition 4 possesses property (B).

According to the Lemma we can conclude that $v(x, b)$ possesses property (B) if $v(0, b) \equiv 0$ and for every ε (no matter how small) there exists b^* for which $\inf[v(x, b^*): \|x\| = \varepsilon] > 0$.

2°. **Some indications of motion stiffness.** Theorem 1. If for system (1.4):

a) there exists a function $v(t, x, b)$, possessing property (A) with respect to x_α;

b) function v and its derivative $v^\cdot$ (by virtue of system (1.4)) satisfy the condition: for every ε_1, δ_2 and $t_0 \geqslant t^*$ specified in advance one can find $a^* \in D$, ε_2 and δ_1 for which $v^\cdot(t, x, a^*) \leqslant 0$ holds together with (2.4) for all $t \geqslant t_0$ and $x \in \overline{\Pi}_\varepsilon$, then the motion $x = 0$ possesses stiffness with respect to x_α.

Proof. Assume that the theorem 's hypotheses are fulfilled: the parameter a^* has been defined and the domains $\overline{\Pi}_\varepsilon$ and $\overline{\Pi}_\delta$ have been constructed for arbitrary ε_1, δ_2 and t_0. Then the solution $x(t, t_0, x_0, a^*) \in \Pi_\varepsilon$ if only $x_0 \in \overline{\Pi}_\delta$. As a matter of fact, arguing otherwise, we assume that when $a = a^*$ a solution $x(t)$ exists reaching the boundary of $\overline{\Pi}_\varepsilon$ at an instant $t_1 > t_0$ notwithstanding that the condition $x_0 \in \overline{\Pi}_\delta$ obtains at $t = t_0$. Since solution $x(t) \in \overline{\Pi}_\varepsilon$ when $t \in [t_0, t_1]$, function v does not grow along it; consequently, $v(t_1, x(t_1), b^*) \leqslant v(t_0, x_0, b^*)$, which contradicts (2.4). The motion $x = 0$ possesses the stiffness property.

Corollary 1. If a function $v(t, x, b)$ possessing property (A) with respect to x_α exists and $v^\cdot(t, x, a) \leqslant 0$ for all $t \geqslant t^*$, $x \in \Gamma$ and $a \in D$, then motion $x = 0$ possesses stiffness with respect to x_α.

Theorem 2. If for system (1.4):

a) there exists a function $v(t, x, b)$ possessing property (A) with respect to x_α uniformly in $t \in [t^*, \infty)$;

b) function v and its derivative $v^\cdot$ satisfy the condition: for every ε_1 and δ_2 specified in advance one can find $a^* \in D$, ε_2 and δ_1 for which $v^\cdot(t, x, a^*) \leqslant 0$ holds together with (2.6) for all $t \geqslant t^*$ and $x \in \overline{\Pi}_\varepsilon \setminus \Pi_\delta$, then the motion $x = 0$ possesses stiffness with respect to x_α uniformly in $t_0 \in [t^*, \infty)$.

Proof. Arguing to the contrary, we assume that when $a = a^*$ a solution $x(t)$ $(x(t_0) = x_0,\ x_0 \in \overline{\Pi}_\delta,\ t_0 \geqslant t^*)$ exists reaching the boundary of $\overline{\Pi}_\varepsilon$ at an instant $t_1 > t_0$. Let t' $(t_0 \leqslant t' < t_1)$ be an instant for which $x(t') \in \overline{\Pi}_\delta \setminus \Pi_\delta$ and let $x(t) \in \overline{\Pi}_\varepsilon \setminus \Pi_\delta$ if $t \in [t', t_1]$. Function v does not grow along solution $x(t)$ on the time interval indicated and, therefore, $v(t_1, x(t_1), b^*) \leqslant v(t', x(t'), b^*)$. The latter contradicts condition (2.6). Consequently, $x(t) \in \Pi_\varepsilon$ if $x_0 \in \overline{\Pi}_\delta$ and $t_0 \geqslant t^*$.

Theorem 3. If for system (1.4):

a) a function $v(t, x, b)$ exists possessing property (A) with respect to x_α and admitting of an infinitesimal upper bound at $x = 0$;

b) function v and its derivative $v^\cdot$ satisfy the condition: for every ε_1, δ_2 and $t_0 \geqslant t^*$ specified in advance we can find $a^* \in D$, ε_2 and δ_1 for which (2.4) is fulfilled and $v(t, x, b^*)$ is positive definite in domain $\overline{\Pi}_\varepsilon$ when $t \geqslant t_0$ while $v^\cdot(t, x, a^*)$ is negative definite, then the motion $x = 0$ possesses strong stiffness with respect to x_α.

Proof. Function v and its derivative $v^\cdot$ satisfy the hypotheses of Theorem 1 and so the motion $x = 0$ possesses stiffness with respect to x_α. Consequently, every solution $x(t, t_0, x_0, a^*) \in \Pi_\varepsilon$ when $t \geqslant t_0$ if $x_0 \in \overline{\Pi}_\delta$. It remains to show that $x(t, t_0, x_0, a^*) \to 0$ as $t \to \infty$. The latter can be established by using, say, the proof scheme of Theorem II in [7].

Corollary 2. If a function $v(t, x, b)$ exists that possesses property (A) with respect to x_α, is positive definite and admits of an infinitesimal upper bound at $x = 0$, while $v^\cdot(t, x, a)$ is negative definite for all $t \geqslant t^*$, $x \in \Gamma$ and $a \in D$, then the motion $x = 0$ possesses strong stiffness with respect to x_α.

Note 2. Theorem 1 can be extended to practical stability if functions possessing property (B) are used.

3°. We consider system (1.4) under constantly acting perturbations

$$dx/dt = X(t, x, a) + \mu R(t, x, a) \quad (\mu = \text{const} > 0) \tag{2.7}$$

We remark that the need for investigating similar systems with a small parameter μ arises, for instance, in the theory of oscillations and in other problems. Besides the usual requirements on the functions $R_s(t, x, a)$ $(s = 1, \ldots, n)$, we shall assume their uniform boundedness in each domain $\overline{\Pi}_\varepsilon \subset \Gamma_0$ from (1.5) when $t \geqslant t^*$.

Motion $x = 0$ possesses stiffness with respect to x_α constantly acting perturbations if for any ε_1, δ_2 and t_0 we can find a^* and μ^* depending on them and ε_2 and δ_1, defining domain (1.6) for which the solution of (2.7) $x(t, t_0, x_0, a^*, \mu^*) \in \Pi_\varepsilon$ $(t \geqslant t_0)$, if only $x_0 \in \overline{\Pi}_\delta$, for any function R_s.

Let us assume that a function $v(t, x, b)$ satisfying the hypotheses of Theorem 2 has been constructed for system (1.4), with the following additions: $v^\cdot(t, x, a^*) < -l$ $(l = \text{const} > 0)$, $x \in \overline{\Pi}_\varepsilon \setminus \Pi_\delta$, $t \geqslant t^*$, and the derivatives $\partial v(t, x, b^*) / \partial x_s$ $(s = 1, \ldots, n)$ are uniformly bounded in domain $\overline{\Pi}_\varepsilon$ when $t \geqslant t^*$). Then the motion $x = 0$ possesses stiffness under constantly acting perturbations.

Indeed, the motion $x = 0$ of system (1.4) possesses stiffness with respect to x_α uniformly in $t_0 \in [t^*, \infty)$. We assume that parameter a^* has been fixed and the domains $\overline{\Pi}_\varepsilon$ and $\overline{\Pi}_\delta$ for which (2.6) holds have been constructed, and also that $v^\cdot(t, x, a^*) < -l$ $(x \in \overline{\Pi}_\varepsilon \setminus \Pi_\delta$, $t \geqslant t^*$. In addition, $|\partial v(t, x, b^*) / \partial x_s| < N$ and $|R_s(t, x, a^*)| < M$. We set up the expression for the derivative of function $v(t, x, b^*)$ by virtue of system (2.7). We obtain

$$v^\cdot(t, x, a^*, \mu)_{(2.7)} = v^\cdot(t, x, a^*) + \mu \sum_{s=1}^{n} \frac{\partial v(t, x, b^*)}{\partial x_s} R_s(t, x, a^*)$$

whence it follows that if $x \in \overline{\Pi}_\varepsilon \setminus \Pi_\delta$, $t \geqslant t^*$, then

$$v^\cdot(t, x, a^*, \mu)_{(2.7)} < -l + \mu nNM$$

and $v^\cdot(t, x, a^*, \mu^*)_{(2.7)} < 0$ for $\mu^* < l / nNM$. The latter signifies that the solution of system (2.7) with initial conditions $x_0 \in \overline{\Pi}_\delta$ and $t_0 \geqslant t^*$ do not leave the domain Π_ε for $t > t_0$.

As we can see the theorems presented are in a known sense the analogs of motion stability theorems. Indications of nonstiff motion, which we do not discuss here, can be established in similar fashion.

Example 2. Stiffness of a vertically rectified Lagrange gyroscope. For the axis z_1 directed vertically upward the unperturbed motion corresponds to the values (1.7) of the variables, while the perturbed motion, to (1.8). Since variable η_3 can be expressed in terms of η_1 and η_2 (see (1.9)), the equations of perturbed motion, depending on parameter r, can be written for the variables ξ_i and η_i $(i = 1, 2)$. Using the Lagrange integrals for the equations of motion, we can write the integrals of perturbed motion

$$v_1 = A(\xi_1^2 + \xi_2^2) - 2mgz\eta_3, \quad v_2 = A(\xi_1\eta_1 + \xi_2\eta_2) - Cr\eta_3 \qquad (2.8)$$
$$(\eta_3 = 1 - (1 - \eta_1^2 - \eta_2^2)^{1/2})$$

here z is the coordinate of the center of gravity.

Let us consider the bundle of integrals (2.8)

$$v(x, r) = v_1 - \lambda v_2 = A(\xi_1^2 + \xi_2^2) - A\lambda(\xi_1\eta_1 + \xi_2\eta_2) + (\lambda Cr - 2mgz)\eta_3 \qquad (2.9)$$

where λ is a constant not determined as yet. Let us show that the function v in (2.9) possesses property (A) with respect to η_1 and η_2, satisfying the hypotheses of the Lemma presented above. For this purpose we make the required constructions, using domains (1.10).

Let $\varepsilon_1 < 1$ and δ_2 be given. Since $v^*(\xi) = A(\xi_1^2 + \xi_2^2)$,

$$M = \sup[v(x, r): \eta_1 = \eta_2 = 0, \ |\xi_1| \leqslant \delta_2, \ |\xi_2| \leqslant \delta_2] = 2A\delta_2^2$$

On the set $\eta_1^2 + \eta_2^2 = \varepsilon_1^2$ the function v has the minimum

$$\min v = (\lambda Cr - 2mgz)(1 - \sqrt{1 - \varepsilon_1^2}) - {}^1/_4 A\lambda^2\varepsilon_1^2 \qquad (2.10)$$

depending on the bundle's parameter λ. Having chosen the magnitude of this parameter from the condition that expression (2.10) be maximum, we obtain

$$\max\min v = C^2r^2(1 - \sqrt{1 - \varepsilon_1^2})^2 / A\varepsilon_1^2 - 2mgz(1 - \sqrt{1 - \varepsilon_1^2}) \qquad (2.11)$$

We require that $\max\min v > M$. In accord with (2.11) we obtain the following condition for choosing the magnitude of parameter r:

$$C^2r^2 > 2A\varepsilon_1^2[A\delta_2^2 + mgz(1 - \sqrt{1 - \varepsilon_1^2})] / (1 - \sqrt{1 - \varepsilon_1^2})^2 \qquad (2.12)$$

Function (2.9) satisfies the last condition of the Lemma. Indeed, $v(x, r) \to +\infty$ as $\xi_1^2 + \xi_2^2 \to \infty$ uniformly in the domain $\eta_1^2 + \eta_2^2 \leqslant \varepsilon_1^2$.
Having now set $r = \omega + \xi_3$ (2.12), where $|\xi_3| \leqslant \delta_2$, we obtain

$$|\omega| > \delta_2 + \sqrt{2A}\varepsilon_1[A\delta_2^2 + mgz(1 - \sqrt{1 - \varepsilon_1^2})]^{1/2} / C(1 - \sqrt{1 - \varepsilon_1^2}) \qquad (2.13)$$

We note that the numbers δ_1 and ε_2 defining for the variables ξ_i and η_i $(i = 1, 2)$ the domains (1.10) depend on ξ_3. Keeping in mind that $|\xi_3| \leqslant \delta_2$, we can take

$\delta_1 = \inf \delta_1(\xi_3)$ and $\varepsilon_2 = \sup \varepsilon_2(\xi_3)$. Thus the function v of (2.9), being an integral of the equations of perturbed motion, possesses property (A) with respect to η_1 and η_2. Then by Corollary 1 we can deduce that motion (1.7) possesses stiffness with respect to these variables.

In concluding the analysis of the example we note that (2.13) becomes (1.13) when $z = 0$. It can also be shown that (2.12) is fulfilled for sufficiently small ε_1 and δ_2 if the stability condition $C^2\omega^2 > 4\,Amgz$ [8] holds. Indeed, assuming that $\delta_2 / \varepsilon_1 \to 0$ as $\varepsilon_1 \to 0$, we get that the limit of the right-hand side of (2.13) equals $4Amgz$.

Example 3. Property of stiffness-equilibrium of a conservative system. The equilibrium position of a system subject to holonomic and stationary constraints is determined by the generalized coordinates $q_i\,(i = 1,\ldots, n)$. We consider the case when the system's potential energy $\Pi = \Pi(q_1,\ldots, q_n, a_1,\ldots, a_r)$ depends upon parameters and we assume that $q_i = 0$ is an isolated equilibrium position for each $a \in D$. We assume that $\Pi(0, a) \equiv 0$. We denote

$$\varphi(\varepsilon, a) = \inf\,[\Pi(q, a) : q \in K_\varepsilon \setminus \overline{K}_\varepsilon], \quad \overline{K}_\varepsilon = \{q: |q_i| \leqslant \varepsilon\}$$

where ε is sufficiently small.

The equilibrium position $q = 0$ possesses stiffness with respect to the coordinates if for any positive ε and N (the first arbitrarily small, the second, arbitrarily large) a parameter $a^* \in D$ depending on them exists for which $\varphi(\varepsilon, a^*) > N$. Indeed, under the assumptions made the system's total energy

$$v(q, q^\cdot, a) = \frac{1}{2}\sum_{i,j=1}^{n} a_{ij}(q)\, q_i^\cdot q_j^\cdot + \Pi(q, a)$$

possesses property (A) with respect to q, having satisfied the Lemma's conditions. Since by virtue of the equations of motion $v^\cdot \equiv 0$, integral v satisfies the hypotheses of Corollary 1.

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